Domino Tilings of a Rectangular Chessboard

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Solutions

Note: there are almost certainly typos in this solutions guide.

1 The Problem

1. Draw a 2 by 2, 3 by 2, and 4 by 3 chessboard. With some trial and error, one can find that there are 2 tilings of the 2 by 2, 3 tilings of the 3 by 2, and 11 tilings of the 4 by 3. Now, verify the formula for the 2 by 2 case by obtaining the following four terms:

$$(4\cos^2\tfrac{\pi}{3}+4\cos^2\tfrac{\pi}{3})^{\frac{1}{4}}(4\cos^2\tfrac{\pi}{3}+4\cos^2\tfrac{2\pi}{3})^{\frac{1}{4}}(4\cos^2\tfrac{2\pi}{3}+4\cos^2\tfrac{\pi}{3})^{\frac{1}{4}}(4\cos^2\tfrac{2\pi}{3}+4\cos^2\tfrac{2\pi}{3})^{\frac{1}{4}}$$

$$=(2^{\frac{1}{4}})^4=2$$

2. Since dominoes cover 2 squares, a domino tiling can only cover an even number of squares. This means that $Z(M,N)=0$ if M and N are odd. Now verify that the formula is correct in this case. Note that the term in which $m=\frac{M+1}{2}$ and $n=\frac{N+1}{2}$ (which are integers when M and N are odd) will be equal to zero, which means that the formula is correct.

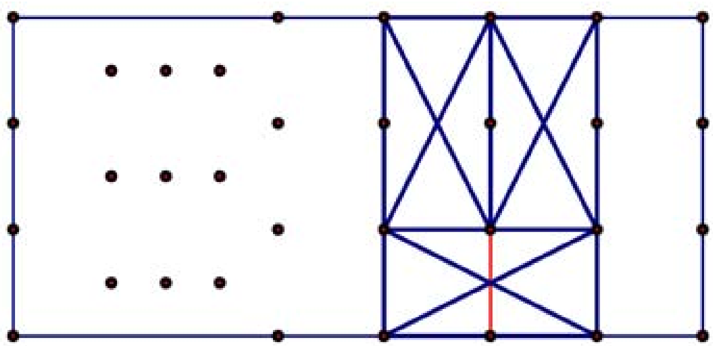
3. When looking at these kinds of problems, try to find a formula based on smaller cases. It is easy to show that $Z(2,1)=1$, $Z(2,2)=2$, $Z(2,3)=3$, and $Z(2,4)=5$. This suggests that the numbers are part of the Fibonacci sequence.

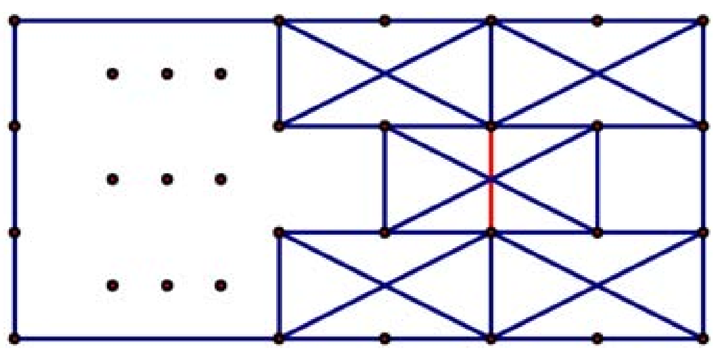
Now let's try to prove that $Z(2,N)=F_N$ where $F_0=F_1=1$. The base cases, $Z(2,1)$ and $Z(2,2)$, have been verified. Now we just need to show that $Z(2,N)=Z(2,N-1)+Z(2,N-2)$. To do this, consider the column of squares farthest to the right. These two squares can either be covered with 1 vertical domino or 2 horizontal dominoes. In the first case, there are $Z(2,N-1)$ ways to tile the rest of the board, as a 2 by $N-1$ board is left for tiling. In the second case, the two horizontal dominoes cover the two columns farthest to the right, leaving an untiled 2 by $N-2$ rectangle, which means that there are $Z(2,N-2)$

ways to tile in this case. Since these two cases do not overlap, the total number of tilings is both $Z(2,N)$ and $Z(2,N-1)+Z(2,N-2)$. These quantities must be equal, so therefore the Fibonacci recursion applies to this situation.

4. Think about this problem in a similar fashion to last problem. It's impossible to tile the column to the far right in the 3 by $2N$ board without a domino covering the column to the left. Therefore, start by considering a tiling that contains dominoes in the 3 by 2 rectangle farthest to the right that do not cover the rest of the rectangle. There are 3 ways to tile the 3 by 2 rectangle and $Z(3,2N-2)$ ways to tile the rest. This means that so far $3Z(3,2N-2)$ tilings have been accounted for.

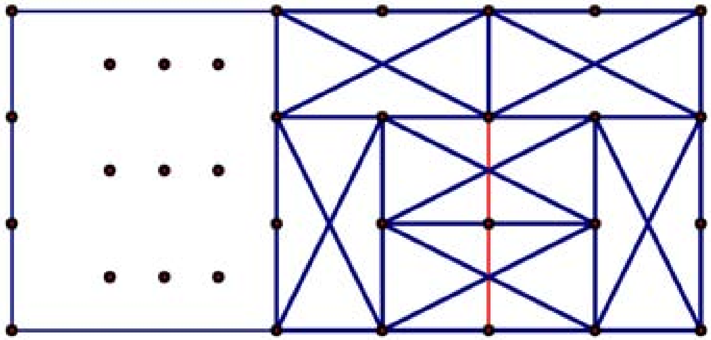
However, we haven't accounted for tilings in which dominoes cover the boundary (represented by a red line) between the 3 by 2 rectangle and the 3 by $2N-2$ rectangle. Now consider the various cases that can occur with dominoes covering that boundary. Suppose that one domino covers the boundary. However, this would result in the following two cases:





In the first situation, the overall rectangle is divided into a 3 by 1 rectangle and a 3 by $2N-3$ rectangle, both of which cannot be tiled by Problem 2. The second case is invalid as the dominoes will never horizontally line up.

Now suppose that there are two dominoes covering the dividing line, as shown below.



The dominoes will line up in this case by placing two vertical dominoes as shown above. There are two ways in which this arrangement can be formed and this arrangement leaves an untiled 3 by $2N-4$ board, so the total number of tilings for this case is $2Z(3,2N-4)$. However, note that the left vertical domino can be replaced by two more horizontal dominoes, resulting in $2Z(3,N-6)$ more cases. This logic continues, resulting in the following recursion:

$$Z(3,2N) = 3Z(3,2N-2) + \sum_{i=0}^{N-2} Z(3,2i)$$

as desired. To obtain the short form of this formula, consider the formulas for $2N$ and $2N-2$:

$$Z(3,2N) = 3Z(3,2N-2) + 2Z(3,2N-4) + 2Z(3,2N-6) + \cdots$$

$$Z(3,2N-2) = 3Z(3,2N-4) + 2Z(3,2N-6) + \cdots$$

Now subtract the second equation from the first. This results in $Z(3,2N) - Z(3,2N-2) = 3Z(3,2N-2) - Z(3,2N-4)$. Adding $Z(3,2N-2)$ results in the stated formula.

2 Matrices, Permutations, and Determinants

1. Switch 4 and 1, 2 and 3, and 5 and 2. This is 3 transpositions, so the sign of the permutation is $(-1)^3 = -1$.

2. When finding the determinant of a matrix, use linearity and alternation to make the overall determinant into a sum of determinants of identity matrices. For simplicity, go from left to right.

Start by noticing that $<1,3,6> = <1,0,0> + 3<0,1,0> + 6<0,0,1>$. By linearity, the desired determinant can be written as:

$$\det\begin{pmatrix}1&2&2\\0&5&7\\0&1&5\end{pmatrix} + 3\det\begin{pmatrix}0&2&2\\1&5&7\\0&1&5\end{pmatrix} + 6\det\begin{pmatrix}0&2&2\\0&5&7\\1&1&5\end{pmatrix}$$

Now use linearity on the second column:

$$(2\det\begin{pmatrix}1&1&2\\0&0&7\\0&0&5\end{pmatrix} + 5\det\begin{pmatrix}1&0&2\\0&1&7\\0&0&5\end{pmatrix} + \det\begin{pmatrix}1&0&2\\0&0&7\\0&1&5\end{pmatrix}) + 3(2\det\begin{pmatrix}0&1&2\\1&0&7\\0&0&5\end{pmatrix} +$$
$$5\det\begin{pmatrix}0&0&2\\1&1&7\\0&0&5\end{pmatrix} + \det\begin{pmatrix}0&0&2\\1&0&7\\0&1&5\end{pmatrix}) + 6(2\det\begin{pmatrix}0&1&2\\0&0&7\\1&0&5\end{pmatrix} + 5\det\begin{pmatrix}0&0&2\\0&1&7\\1&0&5\end{pmatrix} +$$
$$\det\begin{pmatrix}0&0&2\\0&0&7\\1&1&5\end{pmatrix})$$

Eliminate all determinants that are 0 (which have two columns that are the same) and use linearity again:

$$(5(2\det\begin{pmatrix}1&0&1\\0&1&0\\0&0&0\end{pmatrix} + 7\det\begin{pmatrix}1&0&0\\0&1&1\\0&0&0\end{pmatrix} + 5\det\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}) + (2\det\begin{pmatrix}1&0&1\\0&0&0\\0&1&0\end{pmatrix} +$$
$$7\det\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} + 5\det\begin{pmatrix}1&0&0\\0&0&0\\0&1&1\end{pmatrix})) + 3(2(2\det\begin{pmatrix}0&1&1\\1&0&0\\0&0&0\end{pmatrix} + 7\det\begin{pmatrix}0&1&0\\1&0&1\\0&0&0\end{pmatrix} +$$
$$5\det\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}) + (2\det\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} + 7\det\begin{pmatrix}0&0&0\\1&0&1\\0&1&0\end{pmatrix} + 5\det\begin{pmatrix}0&0&0\\1&0&0\\0&1&1\end{pmatrix})) +$$
$$6(2(2\det\begin{pmatrix}0&1&1\\0&0&0\\1&0&0\end{pmatrix} + 7\det\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} + 5\det\begin{pmatrix}0&1&0\\0&0&0\\1&0&1\end{pmatrix}) + 5(2\det\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} +$$
$$7\det\begin{pmatrix}0&0&0\\0&1&1\\1&0&0\end{pmatrix} + 5\det\begin{pmatrix}0&0&0\\0&1&0\\1&0&1\end{pmatrix}))$$

Eliminate all determinants that are 0 (which have two columns that are the same). We are finished using linearity. Now use alternation to turn all of the matrices with ones and zeros into identity matrices. After doing this and using the fact that $\det I = 1$, the determinant comes out to be 5(5)-1(7)-3(2(5))+3(1(2))+6(2(7))-6(5(2))=18.

3. When dealing with permutations, try to switch elements so that one always ends up in its correct place. In this case, start by switching 2 and 1 to put 2 in the first position. Then, switch 3 and 1 to put 3 in the correct place. After continuing this, k elements were moved to the correct place in $k-1$ transpositions (as 1 automatically ends up in the end after these transpositions). This means that the sign of the permutation is $(-1)^{k-1}$. This permutation is called a **k-cycle**.

4. Look back at Problem 2. Notice that each term consists of a product of 3 entries, one from each column, and a matrix with exactly one 1 in each column. This means that for the general n by n matrix, linearity is applied n times, as there are n columns that can be expanded upon. Suppose that T_n represents the set of all n tuples of integers between 1 and n. By generalizing what was found in Problem 2, the determinant can be represented as:

$$\sum_{\sigma \in T_n} M_{1\sigma(1)} \cdots M_{n\sigma(n)} \det(e_{\sigma(1)}|\cdots|e_{\sigma(n)})$$

where $(u|v|w)$ denotes a matrix with 3 columns (the entries of vector u from top to bottom, the entries of vector v, and the entries of the vector w) and e_i denotes a vector with n entries where there is a 1 in the i th entry and 0s in the remaining entries.

Now think back to Problem 2 again. First, we eliminated terms equal to 0, which occur when two columns of $(e_{\sigma(1)}|\cdots|e_{\sigma(n)})$ are the same. Therefore, the sum reduces to:

$$\sum_{\sigma \in S_n} M_{1\sigma(1)} \cdots M_{n\sigma(n)} \det(e_{\sigma(1)}|\cdots|e_{\sigma(n)})$$

where S_n is the set of all permutations acting on $\{1, 2, \cdots, n\}$.

Once again, think back to Problem 2, where we used alternation to make the determinant expressions into determinants of identity matrices. To do this, notice that the location of the 1 in a column i is $\sigma(i)$. Notice that alternation is like a transposition that changes the sign of the determinant. Therefore,

$$\det(e_{\sigma(1)}|\cdots|e_{\sigma(n)}) = sgn(\sigma^{-1}) \det(e_1|e_2|\cdots|e_n)$$

since the permutation σ is applied in reverse to obtain an identity matrix. Substituting $\det(e_1|e_2|\cdots|e_n) = 1$ and $sgn(\sigma) = sgn(\sigma^{-1})$ results in the correct formula, which is:

$$\sum_{\sigma \in S_n} sgn(\sigma) M_{1\sigma(1)} \cdots M_{n\sigma(n)}$$

5. Up to this point, we have been assuming that the sign of a permutation remains the same no matter what transposition path is taken. If you think about it, this is a huge assumption, because there is no obvious reason why a permutation could not be both an even and an odd number of transpositions away from another transposition.

To prove this, let's look at an example of the polynomial listed in the hint; namely for $n = 4$:

$$(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$$

Suppose that x_2 and x_4 are switched, which corresponds to the permutation $\{1, 2, 3, 4\} \to \{1, 4, 3, 2\}$. This results in the polynomial:

$$(x_1 - x_4)(x_1 - x_3)(x_1 - x_2)(x_4 - x_3)(x_4 - x_2)(x_3 - x_2) =$$
$$-(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$$

It seems that the sign of a permutation corresponds to the sign of this polynomial after switching the corresponding variables. Does this hold for larger values of n?

It turns out that it does. Consider what happens when x_i and x_j are switched for $i < j$. This results in $2(j - i - 1) + 1$ terms that are switched. Since this number is odd, all negative signs cancel out except for one. Therefore, a transposition does multiply the sign of this polynomial by -1, which means that the sign function is well-defined.

6. Use the formula in Problem 4 to evaluate this determinant. The determinant is:

$$\sum_{\sigma, \tau \in S_n} (-1)^n sgn(\sigma) sgn(\tau) B_{\sigma(1)1} \cdots B_{\sigma(n)n} B_{1\tau(1)} \cdots B_{n\tau(n)}.$$

$$= (-1)^n (\det B)^2$$

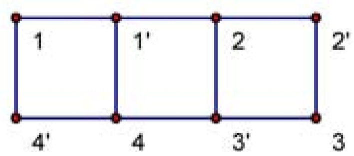
where B is an n by n matrix.

3 Graphs and Adjacency Matrices

1. Label the vertices in a counterclockwise fashion beginning in the upper left hand corner. Then the adjacency matrix is as follows:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

2. Make a 4 by 2 rectangle graph labeled as follows:



The matrix requested is:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

When this determinant is evaluated with the formula in Problem 2.4, the determinant is (-1)+1-1-(-1)-1=-1. Notice that there also are 5 domino tilings of the 4 by 2 chessboard. Is there some way that this matrix could be altered so that its determinant or some function of it gives the number of domino tilings of the chessboard?

3. The matrix requested here is:

$$\begin{pmatrix} 1 & 1 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 1 \\ i & 0 & 0 & 1 \end{pmatrix}$$

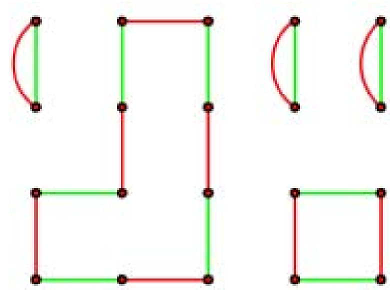
The determinant of this matrix, according to the formula in Problem 2.4, is 5. Notice that the complex absolute value of this determinant is the number of domino tilings of a 4 by 2 chessboard. Why does this adjusted determinant give us the number of tilings?

4. Think about what we're actually doing when we take the determinant of a matrix. By the formula in Problem 2.4, we calculate the determinant by taking the sum of a bunch of terms, where each term is the product of n numbers in and n by n matrix and a positive or negative sign. However, these aren't just any numbers; exactly one element is taken from each row and column. When taking the adjusted the determinant in the previous problem, this means that a term is nonzero if and only if the the term corresponds to a perfect matching. However, go back to Problem 2. Some of the terms canceled out. So in order to prove that the determinant in Problem 3 gives us the number of perfect matchings, we need to prove that the signs of all of the terms in the determinant are the same. In other words, we need to prove that for all $\tau, \sigma \in S_n$ that:

$$sgn(\tau)M_{1\tau(1)} \cdots M_{n\tau(n)} = sgn(\sigma)M_{1\sigma(1)} \cdots M_{n\sigma(n)}$$

where σ and τ are permutations that represent perfect matchings, in other words n is connected to $\sigma(n)'$ in the first matching and $\tau(n)'$ in the second.

Consider what happens when you overlay two perfect matchings in the same area. Since each vertex is connected to exactly one other vertex in each perfect matching, each vertex will be connected to exactly two not necessarily distinct vertices in the overlay graph. However, this means that the overlay graph either is composed of double edges between points or circuits. The two perfect matchings only differ along the circuit.

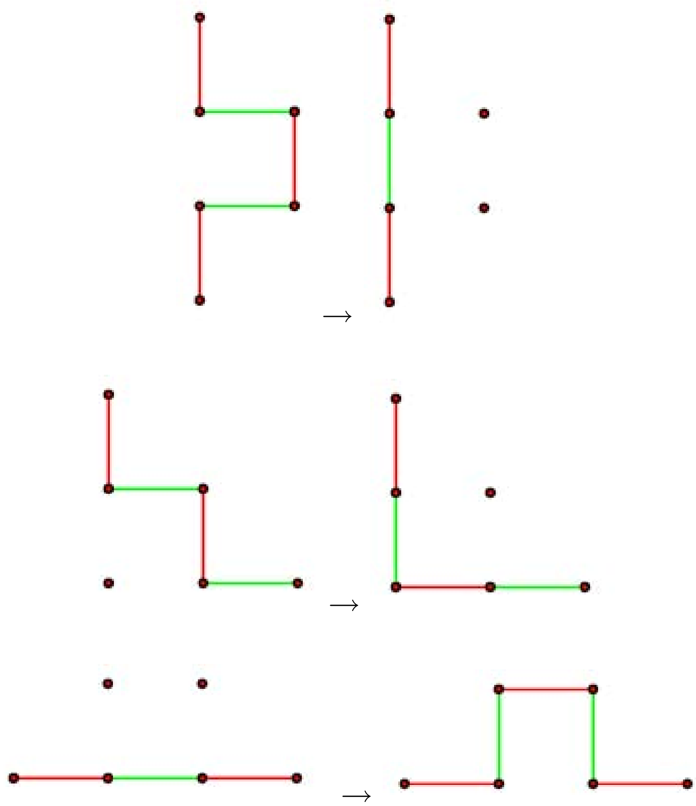


Suppose that an overlay of two perfect matchings contains only one circuit passing through $2k$ points. In the above diagram, note that the part of τ that corresponds to the circuit above is obtained by composing σ with a k-cycle. Therefore,

$$sgn(\tau) = sgn(\sigma)(-1)^{k-1}$$

by Problem 2.3. To finish this problem, we just need to show that $M_{1\tau(1)} \cdots M_{n\tau(n)} = (-1)^{k-1} M_{1\sigma(1)} \cdots M_{n\sigma(n)}$.

To do this, see what happens to each side when you chop off a square:



The sign always changes by -1. Therefore, the difference should just be $(-1)^A$ where A is the area contained by the circuit. The area of this polygon can be found with Pick's Theorem, which states that if a polygon with vertices at lattice points (points with integer coordinates) contains I interior lattice points and B lattice points on the boundary, the area of the polygon is $I + \frac{B}{2} - 1$. Notice that I must be even, as all of the points within the polygon must be matched to another point within the polygon. Also, notice that $B = 2k$. Therefore, the area of the polygon is $even + k - 1$. Since $(-1)^{even} = 1$, the sign changes by $(-1)^{k-1}$ as desired. Therefore, the weighting proposed in Problem 3 works, as it ensures that the sign of all terms is the same. QED

4 Eigenvalues and Eigenvectors

1. By definition λ (a number) is an eigenvalue if there is a vector v such that $Av = \lambda v$. Suppose that $v = < x, y >$. Now convert this matrix equation into a system of two equations:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow x + y = \lambda x \text{ and } x = \lambda y$$

Substitute the second equation into the first to obtain $\lambda y + y = \lambda^2 y$. Divide by y since $y \neq 0$. Solving for λ results in $\lambda = \frac{1 \pm \sqrt{5}}{2}$ as the two eigenvalues.

Now, use $x = \lambda y$ to produce eigenvectors. In particular, the valid eigenvectors $< x, y >$ are $< x, x\frac{1+\sqrt{5}}{2} >$ and $< x, x\frac{1-\sqrt{5}}{2} >$.

2. Fibonacci numbers are obtained by taking exponents of this matrix. Therefore, the eigenvalues of this matrix can be used to find an explicit formula for the Fibonacci numbers.

3. Consider the equation $Av = \lambda v$. Currently, we have used a method of finding eigenvalues that is quite annoying for large matrices. Maybe, we could find the eigenvalues in a more systematic way. Since there are **v**s on both sides of the equation, try subtracting on both sides:

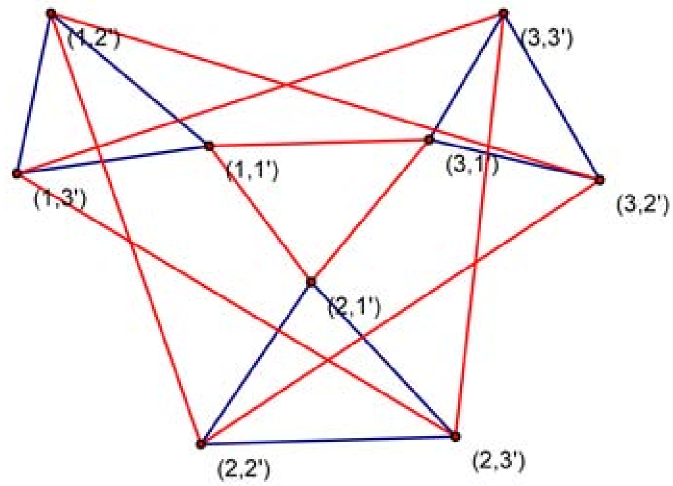
$$Av - \lambda v = Av - \lambda I v = (A - \lambda I)v = 0.$$

where I is the identity matrix with the same dimensions as A. Now, consider when this equation has nonzero solutions v (as the zero vector is not an eigenvector). If we could find the inverse of $A - \lambda I$, then we could show that the zero vector is the only solution for $\mathbf{v}$. Therefore, there are eigenvectors for a certain eigenvalue λ if and only if $\det(A - \lambda I) = 0$ (as matrices are invertible if and only if their determinants are not 0). The determinant on the left side will produce a polynomial in λ. The product of the roots of any polynomial is $\pm$ the constant term of the polynomial. However, the constant term is $\det A$, so the product of the eigenvalues is the determinant. QED

5 Cartesian Products of Graphs

1. The Cartesian product of two triangles will consist of 9 vertices labeled $(1, 1'), (1, 2'), (1, 3'),$ $(2, 1'), (2, 2'), (2, 3'), (3, 1'), (3, 2'),$ and $(3, 3')$. There are many different ways to place the ver-

tices. However, the graph should look something like this:



where vertices with one coordinate in common and the other coordinates connected are connected in the Cartesian product graph.

2. The simplest way to do this problem is by brute forcing it with algebra, so I will not present the solution here.

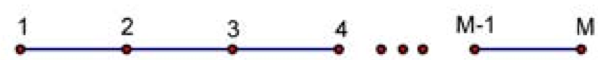
3. First, think about approximately what the adjacency matrix of the Cartesian product of two graphs G and H should look like. Suppose that G has g vertices labeled $1, 2, \cdots, g$ and H has h vertices labeled $1', 2', \cdots, h'$. Since $G\Box H$ has vertices formed by ordered pairs of one vertex from G and one from H, $G\Box H$ must contain gh vertices and the adjacency matrix of $G\Box H$ will be a gh by gh matrix.

Now, we need to figure out how to most conveniently label the rows and columns of this adjacency matrix. Notice that each row can be divided into g blocks of h entries since each row contains gh entries. Therefore, it seems logical that each block of h entries should represent ordered pairs with the same vertex from graph G. Consider the position of (i, j') within a row for $0 < i \leq g$ and $0 < j \leq h$. $i - 1$ blocks of g elements were passed to get to the i^{th} block and this vertex is the j^{th} element within this block, so therefore it makes sense to put the adjacency information for vertex (i, j') in the $((i-1)g+j)$th column. There is no reason to label the rows differently, so therefore insert a 1 for the entry in the $((a-1)g+b)$th row and $((i-1)g+j)$th column if vertices (i, j') and (a, b') are connected; otherwise insert a 0.

Now that we have labeled the adjacency matrix of $G\Box H$, we can show that $A_{G\Box H} = A_G \otimes I_h + I_g \otimes A_H$. First, intepret $A_G \otimes I_h$. The entry in the $((a-1)g+b)$th row and $((i-1)g+j)$th column of this matrix will be the product of the elements in the ath row and ith column of A_G and the bth row and jth column of I_h by the definition of the tensor product (see lecture notes). In other words, the entry in consideration is 1 if both vertices a and i are connected in G and $b = j$. Now, interpret $I_g \otimes A_H$. By a similar argument, this represents the case where $a = i$ and vertex b' is connected to vertex j' in H. Therefore, the sum of these matrices accounts for the overall adjacency matrix of $A_{G\Box H}$. Note that no entry in the overall adjacency matrix can be 2 since that would require two vertices in G to be connected and the same, which cannot happen.

6 Tying it all together

1. As always, we need to start by labeling the vertices of a $1 \times M$ row graph and the rows and columns of the adjacency matrix. Label the graph as shown below



so that the $M \times M$ adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Suppose that $\mathbf{x}$=$x_1, x_2, \cdots, x_M$ is an eigenvector and λ is the eigenvalue for that eigenvector. Multiplying the adjacency matrix by the eigenvector gives a system of equations:

$x_2 = \lambda x_1$

$x_1 + x_3 = \lambda x_2$

$\vdots$

$x_{M-2} + x_M = \lambda x_{M-1}$

$x_{M-1} = \lambda x_M$

Now, think about how the hint applies to this system of equations. If we let $x_k = \sin k\theta$ for some angle θ, then we see that $\lambda = 2\cos\theta$ applies for every equation except for the last one. Now, we need to find the value of θ that would make the last equation true. Letting $k = M$ results in $\sin(M-1)\theta + \sin(M+1)\theta = 2\cos\theta \sin M\theta$. In order for this equation to be the same as the last equation, we need $\sin(M+1)\theta = 0$. This occurs when $\theta = \frac{m\pi}{M+1}$ for all integers m. This means that the eigenvalues are of the form $2\cos\frac{m\pi}{M+1}$. Letting m be any integer between 1 and M inclusive produces M distinct eigenvalues. Since we have found M eigenvalues for this matrix and have found a corresponding eigenvector for each one (just plug in $\theta = \frac{m\pi}{}$ for $x_k = \sin k\theta$) we're done.

2. Look at Problem 5.3. Suppose that $\mathbf{u}$ is an eigenvector of A_G with eigenvalue μ and $\mathbf{v}$ is an eigenvector of A_H with eigenvalue λ. Consider for a moment what happens when we compute

$$A_{G\Box H}(\mathbf{u}\otimes\mathbf{v})$$

$$= A_G \otimes I_h(\mathbf{u}\otimes\mathbf{v}) + I_g \otimes A_H(\mathbf{u}\otimes\mathbf{v})$$

$$= (A_G\mathbf{u}) \otimes (I_h\mathbf{v}) + (I_g\mathbf{u}) \otimes (A_H\mathbf{v})$$

$$= (\mu + \lambda)(\mathbf{u}\otimes\mathbf{v})$$

Therefore, $\mathbf{u}\otimes\mathbf{v}$ is an eigenvector of $A_{G\Box H}$. This means that the eigenvalues of $A_{G\Box H}$ are the pairwise sums of eigenvalues of A_G and A_H. Note that in the original adjacency matrix the entries of A_H (the vertical column $1 \times N$ graph) were weighted by i. Therefore, the eigenvalues will also be weighted by i and its eigenvalues are of the form $2i\cos\frac{n\pi}{N+1}$. Therefore, the eigenvalues of the weighted adjacency matrix $A_{G\Box H}$ are $2\cos\frac{m\pi}{M+1} + 2i\cos\frac{n\pi}{N+1}$.

3. From Problems 3.3 and 2.6, we need to find $\sqrt{|\det A_{G\Box H}|}$ in order to find $Z(M,N)$ since the adjacency matrix of a bipartite graph will always be of the form shown in Problem 2.6. Since the determinant of a matrix is the product of its eigenvalues, we know that

$$|\det A_{G\Box H}| = \prod_{m=1}^{M}\prod_{n=1}^{N} |2\cos\frac{m\pi}{M+1} + 2i\cos\frac{n\pi}{N+1}|$$

$$= \prod_{m=1}^{M}\prod_{n=1}^{N} (4\cos^2\frac{m\pi}{M+1} + 4\cos^2\frac{n\pi}{N+1})^{\frac{1}{2}}$$

Taking the square root of this quantity gives $Z(M, N)$, which is:

$$\prod_{m=1}^{M}\prod_{n=1}^{N}(4\cos^2\frac{m\pi}{M+1}+4\cos^2\frac{n\pi}{N+1})^{\frac{1}{4}}$$